Online Appendix to "Invoicing Currency and Optimal Policies in a Global Liquidity Trap"

A.1 Linear-Quadratic Framework

This section develops the foundation for our optimal policy analysis by presenting the model's log-linear equilibrium conditions and deriving quadratic welfare loss functions under each pricing regime. We proceed in two steps: first characterizing the linear constraints that govern private sector behavior, then deriving the welfare functions that guide optimal policy design.

A.1.1 Linear constraint

We first characterize the log-linear equilibrium conditions of private agents, which serve as constraints for optimal policy problem. For any variable X_t, we define two key deviations: $\hat{X}_t \equiv \log(X_t/X)$ and $\tilde{X}_t \equiv \log(X_t/X_t^{fb})$, where X and X_t^{fb} are the efficient steady state of X_t and its efficient level, respectively. We define efficient equilibrium as the equilibrium under flexible prices, no monopolistic competition, complete international financial markets, and where governments choose an optimal fiscal spending rule determined by $V'(N_t) = J'(G_t)$ which is financed by lump-sum taxes.[1] Following Engel (2011) and Cook and Devereux (2013), we express variables as the world ($\hat{X}_t^W$) and the relative ($\hat{X}_t^R$) terms. The world variables are the summation of the corresponding Home and Foreign variables ($\frac{\hat{X}_t+\hat{X}_t^*}{2}$ or $\frac{\hat{X}_{H,t}+\hat{X}_{F,t}}{2}$), and the relative variables are the subtraction between them ($\frac{\hat{X}_t-\hat{X}_t^*}{2}$ or $\frac{\hat{X}_{H,t}-\hat{X}_{F,t}}{2}$).

The demand block of our log-linear system consists of world and relative IS equations, derived by combining household's consumption Euler equations, risk-sharing conditions, and the market clearing conditions. We first examine how global variables evolve under symmetric pricing regimes (PCP and LCP), then analyze cross-country transmission patterns that differ markedly across regimes.

Under symmetric pricing regimes (PCP and LCP), the world IS curve takes the form:

$$\frac{1}{c_y}\left(\tilde{Y}_t^W - E_t\tilde{Y}_{t+1}^W\right) - \frac{1-c_y}{c_y}\left(\tilde{G}_t^W - E_t\tilde{G}_{t+1}^W\right) = -\frac{1}{\rho}\left(\frac{\hat{R}_t + \hat{R}_t^*}{2} - E_t\hat{\pi}_{t+1}^W - \frac{\hat{R}_{ppi,t}^{fb} + \hat{R}_{ppi,t}^{fb,*}}{2}\right), \quad \text{(A1)}$$

where $\hat{R}_{ppi,t}^{fb}$ denote PPI-based real interest rate prevailing in the efficient equilibrium and $c_y = \frac{C}{Y}$ is the steady-state consumption-to-output ratio. This equation shows how global output responds to average interest rates and fiscal policy. Notably, this relationship remains identical across PCP and LCP regimes when preferences and technology are symmetric across countries.

Meanwhile, the degree of ERPT significantly influences how shocks transmit between countries, creating distinct patterns in relative economic outcomes. To understand these cross-country differences, we analyze how relative output dynamics respond to policy and economic conditions

[1] Since we impose steady state subsidy, which eliminates distortions from the monopolistic competition, the model steady state is equivalent to the efficient steady state (i.e., $X = X^{fb}$).

under each regime. The relative output dynamics under PCP follow:

$$\frac{1}{c_y D}\left(\tilde{Y}_t^R - E_t\tilde{Y}_{t+1}^R\right) - \frac{(1-c_y)}{c_y D}\left(\tilde{G}_t^R - E_t\tilde{G}_{t+1}^R\right) = -\frac{1}{\rho}\left(\frac{\hat{R}_t - \hat{R}_t^*}{2} - E_t\hat{\pi}_{ppi,t+1}^R - \frac{\hat{R}_{ppi,t}^{fb} - \hat{R}_{ppi,t}^{fb,*}}{2}\right) \tag{A2}$$

where $D = (2\nu-1)^2 + 4\rho\theta\nu(1-\nu)$. When firms instead set prices in the destination market's currency (LCP), imperfect pass-through fundamentally alters these relative dynamics:

$$\begin{aligned}\frac{2\nu-1}{c_y D}\left(\tilde{Y}_t^R - E_t\tilde{Y}_{t+1}^R\right) - \frac{(2\nu-1)(1-c_y)}{c_y D}\left(\tilde{G}_t^R - E_t\tilde{G}_{t+1}^R\right) + \frac{2\nu(1-\nu)\theta}{D}\left(\hat{m}_t - E_t\hat{m}_{t+1}\right) \\ = -\frac{1}{\rho}\left(\frac{\hat{R}_t - \hat{R}_t^*}{2} - E_t\hat{\pi}_{cpi,t+1}^R - \frac{\hat{R}_{cpi,t}^{fb} - \hat{R}_{cpi,t}^{fb,*}}{2}\right),\end{aligned} \tag{A3}$$

where $\hat{R}_{cpi,t}^{fb}$ denotes the CPI-based natural real interest rates. The shift from PPI to CPI-based rates reflects how the welfare-relevant measure of inflation changes under LCP: when export prices are sticky in local currency, consumer prices become the appropriate gauge for measuring real interest rates and welfare effects. Comparing (A2) and (A3), the key distinction under LCP is the emergence of currency misalignment ($\hat{m}_t$), which measures deviations from the law of one price due to sticky export prices. This misalignment creates an additional channel affecting relative dynamics: when the home currency depreciates ($\hat{m}_t > 0$), sticky export prices limit the increase in Foreign import demand, leading to a smaller relative output gap improvement than under PCP. Conversely, when the home currency appreciates ($\hat{m}_t < 0$), export price stickiness moderates the decline in Foreign import demand, providing insulation against adverse exchange rate movements.

The supply side of our model is characterized by New Keynesian Phillips Curves (NKPC) that govern inflation dynamics. Under symmetric pricing regimes, the world NKPC follows:

$$\hat{\pi}_t^W = \kappa\left(\left(\frac{\rho}{c_y} + \eta\right)\tilde{Y}_t^W - \frac{\rho(1-c_y)}{c_y}\tilde{G}_t^W\right) + \beta E_t\hat{\pi}_{t+1}^W, \tag{A4}$$

where $\kappa = \frac{(1-\alpha\beta)(1-\alpha)}{\alpha}$ represents the slope of the Phillips curve. This equation links world inflation to global output and fiscal policy. Like the world IS curve, this relationship remains identical across PCP and LCP regimes, reflecting how the degree of ERPT affects only relative, not global, inflation dynamics.

The relative inflation dynamics, however, differ markedly across pricing regimes, mirroring the patterns we observed in the demand block. Under PCP, relative inflation follows:

$$\hat{\pi}_{ppi,t}^R = \kappa\left(\left(\frac{\rho}{c_y D} + \eta\right)\tilde{Y}_t^R - \frac{\rho(1-c_y)}{c_y D}\tilde{G}_t^R\right) + \beta E_t\hat{\pi}_{ppi,t+1}^R, \tag{A5}$$

where the relative NKPC is expressed in terms of relative PPI inflation ($\hat{\pi}_{ppi,t}^R = \frac{\hat{\pi}_{H,t} - \hat{\pi}_{F,t}^*}{2}$). Under LCP, sticky export prices introduce additional complexity to relative inflation dynamics:

$$\begin{aligned}\hat{\pi}_{cpi,t}^R = &\kappa\left((2\nu-1)\left(\frac{\rho}{c_y D} + \eta\right)\tilde{Y}_t^R - (2\nu-1)\frac{\rho(1-c_y)}{c_y D}\tilde{G}_t^R\right. \\ &\left. + \frac{D-(2\nu-1)^2}{2D}\hat{m}_t\right) + \beta E_t\hat{\pi}_{cpi,t+1}^R,\end{aligned} \tag{A6}$$

where it is now in terms of relative CPI inflation ($\hat{\pi}_{cpi,t}^R = \frac{\hat{\pi}_t - \hat{\pi}_t^*}{2}$) under LCP. Similar to the demand block, currency misalignment influences relative CPI dynamics under LCP. Specifically, when the

Home currency depreciates ($\hat{m}_t > 0$), incomplete ERPT makes imported goods relatively less expensive at Home, stimulating consumption compared to a scenario of complete pass-through. This wealth effect, in turn, reduces labor supply compared to PCP in Home, contributing to a higher cross-country CPI inflation differential.

One notable feature of the symmetric pricing regime is that the global dimension of the model can be fully characterized by the World IS curve (A1) and World NKPC (A4). These equations determine the paths of world output gap ($\tilde{Y}_t^W$) and inflation ($\hat{\pi}_t^W$). Thus, when countries implement identical policy paths $\left(\{\hat{R}_t^W\}_{t=0}^{\infty}, \{\hat{G}_t^W\}_{t=0}^{\infty}\right)$, world economic outcomes are identical across two symmetric pricing regimes. Hence, the extent to which export prices reflect exchange rate movements only affects cross-country differential in output gap and inflation while keeping world variables intact.

While the IS curves and NKPCs alone constitute the complete set of linear constraints under PCP, sluggish relative price dynamics and resulting currency misalignment introduce additional constraints for optimal policy analysis under LCP. Equation (A7) and (A8) describes the dynamics of relative price[2] and determination of currency misalignment under LCP. Unlike PCP, relative price follows the second-order linear difference equation, which leads to history-dependent sluggish adjustment. Since only Home exporting firms price goods in local currency, only foreign relative price (T_t^*) exhibits inertia compared to symmetric LCP.

$$\textbf{\textit{LCP:}} \qquad \tilde{T}_t = \frac{1}{c_y}\frac{2\rho}{D}\tilde{Y}_t^R - \frac{1-c_y}{c_y}\frac{2\rho}{D}\tilde{G}_t^R - \frac{2\nu-1}{D}\hat{m}_t \tag{A7}$$

$$\Delta\hat{T}_t = \beta E_t \Delta\hat{T}_{t+1} - \kappa\left(\tilde{T}_t + 2\eta\tilde{Y}_t^R\right) \tag{A8}$$

A.1.2 Quadratic Welfare Loss Function

In this subsection, we present a second-order approximation of the welfare function under cooperation. The joint welfare function is defined as the weighted sum of utilities of Home and Foreign representative households. The second-order approximation of the joint welfare function, defined below, yields quadratic welfare loss functions for PCP (A9), LCP (A10)

$$W = W_H + W_F = E_0\sum_{t=0}^{\infty}\beta^t\left[\frac{1}{2}\left(Z_t\frac{C_t^{1-\rho}}{1-\rho} - \frac{N_t^{1+\eta}}{1+\eta} + \Psi\frac{G_t^{1-\rho}}{1-\rho}\right) + \frac{1}{2}\left(Z_t^*\frac{(C_t^*)^{1-\rho}}{1-\rho} - \frac{(N_t^*)^{1+\eta}}{1+\eta} + \Psi\frac{(G_t^*)^{1-\rho}}{1-\rho}\right)\right]$$

For all quadratic welfare functions, the first six terms are the same, which contain output gaps and fiscal gaps. These terms capture the welfare loss from endogenous markup fluctuation incurred by price stickiness and deviation from the Samuelson condition of public good provision. The most salient difference in stabilization objectives across pricing regimes is the inflation rate. Under PCP, where export price directly reflects exchange rate movements, stabilizing PPI inflation is equivalent to eliminating domestic and import price dispersions simultaneously. Hence, the welfare loss function only contains world and relative PPI inflations. However, under LCP, the rigidity of export prices causes independent price dispersion in imported goods, shifting the social planner's stabilization objective toward CPI inflation. Under asymmetric pricing regimes where only foreign export prices feature stickiness, stabilization objectives become world and relative PPI inflation with CPI inflation of Foreign. Finally, in LCP, the squared term of currency

[2] Due to symmetry, tracing home relative price dynamics suffices to characterize the full set of private equilibrium conditions.

misalignment represents the social welfare loss caused by relative consumption distortions due to incomplete pass-through.

$$
\begin{aligned}
\textbf{\textit{PCP:}} \quad 2\left\{\mathcal{L}_t^{W,PCP}-(\mathcal{L}_t^W)^{fb}\right\} =& -\left(\frac{\rho}{c_y}+\eta\right)\left(\tilde{Y}_t^W\right)^2-\left(\frac{\rho}{c_yD}+\eta\right)\left(\tilde{Y}_t^R\right)^2 \\
& -\rho\frac{1-c_y}{c_y}\left(\tilde{G}_t^W\right)^2-\rho(1-c_y)\left(\frac{1-c_y}{c_yD}+1\right)\left(\tilde{G}_t^R\right)^2 \\
& +2\rho\frac{1-c_y}{c_y}\tilde{Y}_t^W\tilde{G}_t^W+2\rho\frac{1-c_y}{c_yD}\tilde{Y}_t^R\tilde{G}_t^R \\
& -\frac{\sigma}{\kappa}\left((\hat{\pi}_t^W)^2+(\hat{\pi}_{ppi,t}^R)^2\right), \qquad \text{(A9)} \\
\textbf{\textit{LCP:}} \quad 2\left\{\mathcal{L}_t^{W,LCP}-(\mathcal{L}_t^W)^{fb}\right\} =& -\left(\frac{\rho}{c_y}+\eta\right)\left(\tilde{Y}_t^W\right)^2-\left(\frac{\rho}{c_yD}+\eta\right)\left(\tilde{Y}_t^R\right)^2 \\
& -\rho\frac{1-c_y}{c_y}\left(\tilde{G}_t^W\right)^2-\rho(1-c_y)\left(\frac{1-c_y}{c_yD}+1\right)\left(\tilde{G}_t^R\right)^2 \\
& +2\rho\frac{1-c_y}{c_y}\tilde{Y}_t^W\tilde{G}_t^W+2\rho\frac{1-c_y}{c_yD}\tilde{Y}_t^R\tilde{G}_t^R \\
& -\frac{c_y\theta\nu(1-\nu)}{D}\hat{m}_t^2 \\
& -\frac{\sigma}{\kappa}\left(\left(\hat{\pi}_t^W\right)^2+\left(\hat{\pi}_{cpi,t}^R\right)^2-2\left(1-c_y\right)\left(1-v\right)\hat{\pi}_{cpi,t}^R\Delta\hat{T}_t\right. \\
& \left.+\left(c_y(2v-1)+(1-v)\right)\left(1-v\right)\left(\Delta\hat{T}_t\right)^2\right), \qquad \text{(A10)}
\end{aligned}
$$

To sum up, under PCP, the quadratic welfare function (A9) and four linear constraints of (A1), (A2), (A4), (A5) constitute the linear-quadratic optimal policy problem. Under LCP, the quadratic welfare function (A10) and six constraints of (A1), (A3), (A4), (A6), (A7), and (A8) become the ingredients of optimal policy problem.

A.2 Robustness Checks

We demonstrate that our key findings about the relationship between pricing regimes and optimal policy responses remain robust across different parameter specifications. We examine variations in three additional critical parameters from our baseline calibration: trade elasticity θ, inverse elasticity of intertemporal substitution ρ, and Calvo price stickiness α. These parameters have been identified as fundamental determinants of international shock transmission and policy effectiveness in the open economy literature (Benigno and Benigno, 2006; Corsetti et al., 2011). For each alternative parameter configuration, we analyze how optimal policy varies across both trade openness (ν) and shock magnitudes (σ_z), maintaining the same parameter space explored in Section 7 of the paper. This approach allows us to systematically verify that our core findings about the interaction between pricing regimes, trade integration, and shock size remain qualitatively robust to different parameter choices.

A particularly important composite parameter in open economy models is $\rho\theta$, which governs the substitutability between home and foreign goods. We systematically explore how optimal policy varies under three scenarios: when goods are Edgeworth substitutes ($\rho\theta > 1$), complements ($\rho\theta < 1$), and independent ($\rho\theta = 1$). We achieve these different scenarios through targeted adjustments to either ρ or θ while holding other parameters constant.

Figures A1 and A2 examine the case where $\rho = 0.5$, corresponding to higher intertemporal elasticity of substitution. Under this parameterization, home and foreign goods become Edgeworth independent ($\rho\theta = 1$). This independence weakens the efficacy of Foreign's monetary policy in facilitating expenditure switching or counteracting perverse exchange rate movements under large shocks. Consequently, Foreign maintains rates near zero even under PCP across a broader range of parameters (Figure A1). The higher intertemporal elasticity of substitution also makes consumption more responsive to real interest rate deviations, leading to more severe recessions. This necessitates substantially larger fiscal interventions in both world and relative terms (Figure A2), though the qualitative patterns of optimal policy remain consistent with our baseline findings.

Figures A3 and A4 consider the case with $\rho = 2$, featuring stronger Edgeworth substitutability between goods. While Foreign still raises rates under PCP to manage relative prices (Figures A3), these increases are less pronounced than in our baseline calibration. This moderated monetary response reflects the lower intertemporal elasticity of substitution, which dampens the severity of recessions and reduces the need for aggressive policy intervention. The world and relative fiscal gaps demonstrate similar qualitative patterns but smaller magnitudes, consistent with this reduced severity (Figure A4).

Figures A5 and A6 analyze the complementarity case by setting $\theta = 0.5$. Under Edgeworth complementarity, both PCP and LCP frameworks exhibit similar monetary policy responses, with Foreign maintaining zero rates across most parameter combinations (Figure A5). This convergence occurs because complementarity weakens the expenditure-switching channel: households consume home and foreign goods together rather than substituting between them. This both reduces the need to manage relative distortions and dampens global recession severity, explaining the smaller world and relative fiscal gaps compared to our baseline (Figure A6).

Figures A7 and A8 examine another case of Edgeworth independence by setting $\theta = 1$. While independence again reduces Foreign's incentive to raise rates under PCP (Figure A7), baseline intertemporal elasticity results in milder recessions than the $\rho = 0.5$ case, requiring smaller fiscal interventions (Figure A8).

Finally, Figures A9 and A10 examine the impact of reduced price stickiness ($\alpha = 0.75$). While the qualitative patterns of optimal monetary and fiscal policy remain unchanged, the magnitude of required interventions increases substantially. This amplification reflects the "Paradox of Flexibility" identified by Eggertsson and Woodford (2003) and Werning (2011), where greater price flexibility can exacerbate economic distortions at the zero lower bound by intensifying deflationary pressures. As can be seen from these robustness exercises, we found that the relationship between pricing regimes and optimal policy responses remains robust across different parameter specifications.

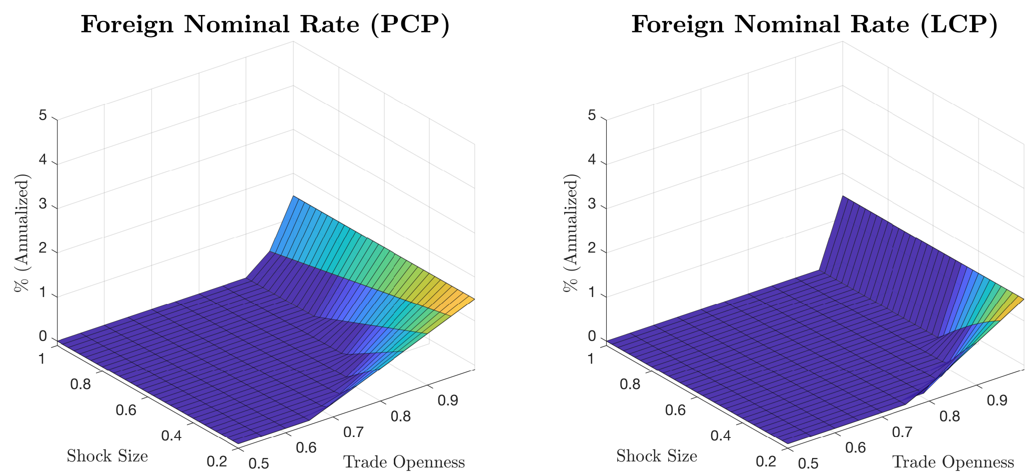

Figure A1: Trade Openness and Optimal Monetary Policy ($\rho = 0.5$)

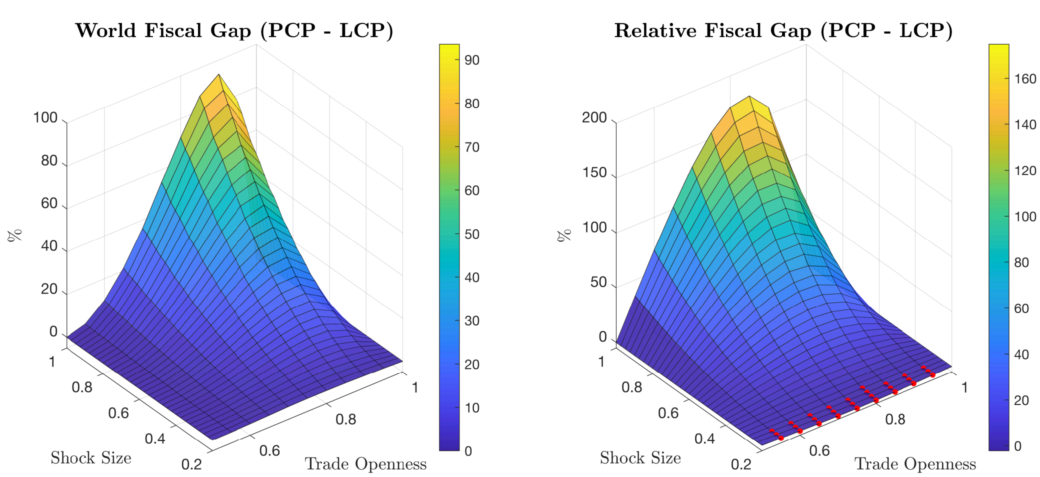

Figure A2: Trade Openness and Optimal Fiscal Policy ($\rho = 0.5$)

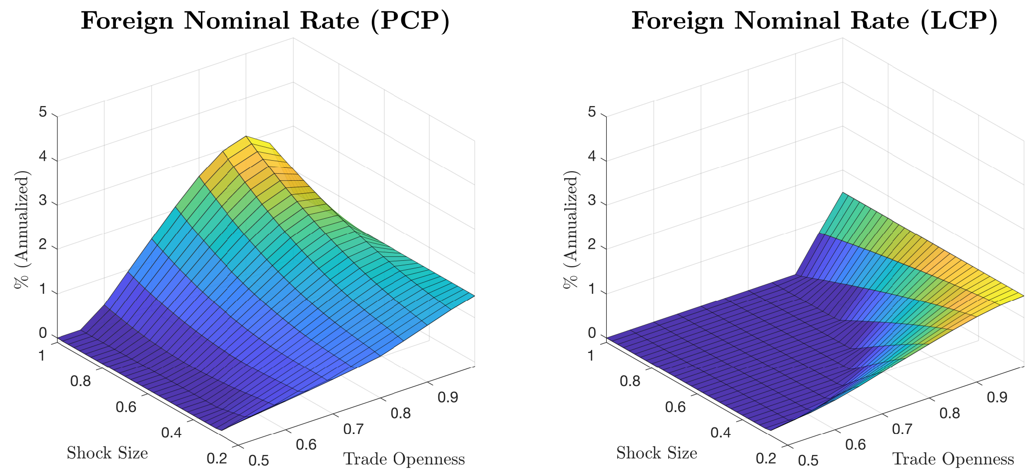

Figure A3: Trade Openness and Optimal Monetary Policy ($\rho = 2$)

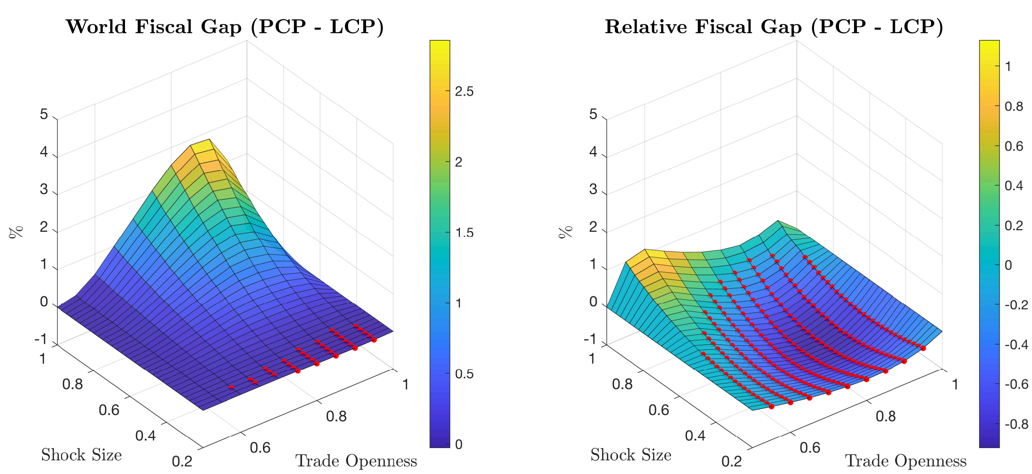

Figure A4: Trade Openness and Optimal Fiscal Policy ($\rho = 2$)

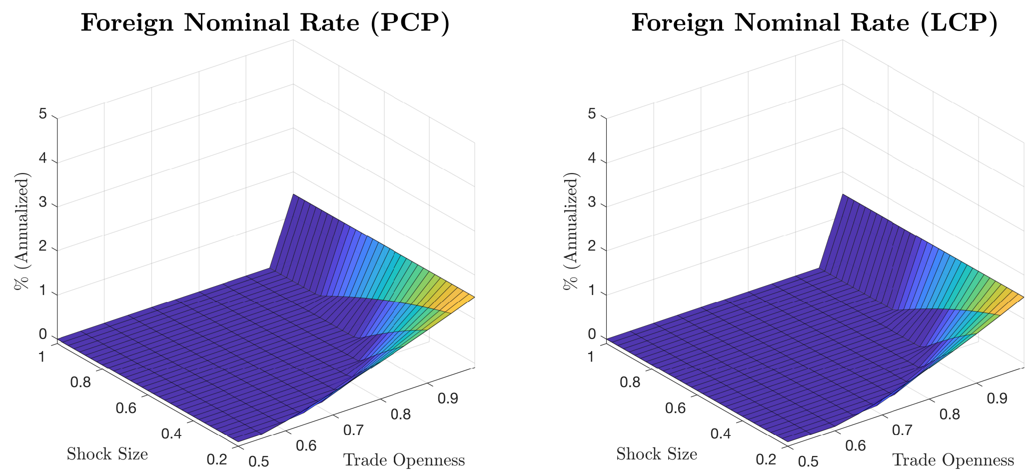

Figure A5: Trade Openness and Optimal Monetary Policy ($\theta = 0.5$)

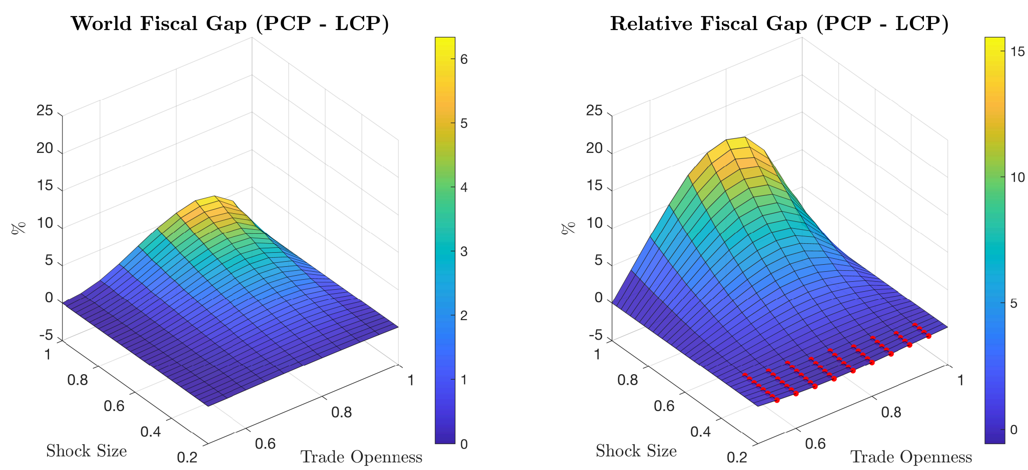

Figure A6: Trade Openness and Optimal Fiscal Policy ($\theta = 0.5$)

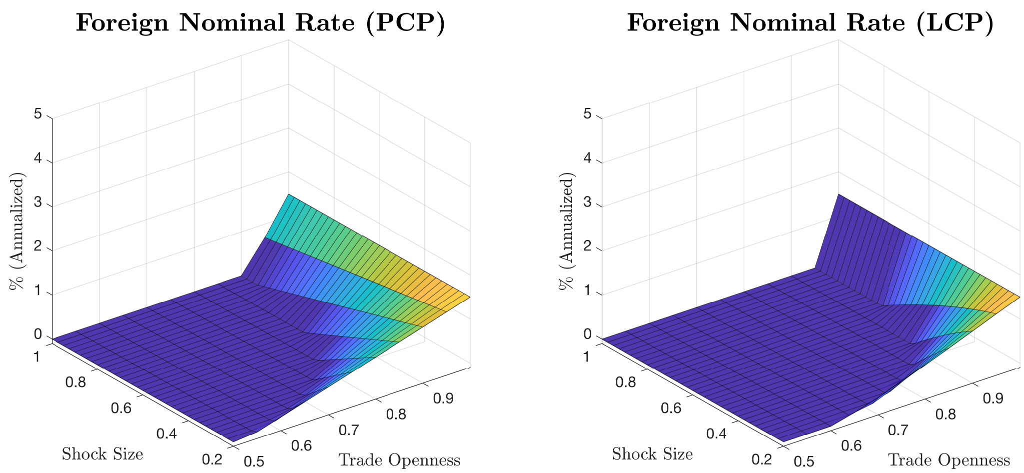

Figure A7: Trade Openness and Optimal Monetary Policy ($\theta = 1$)

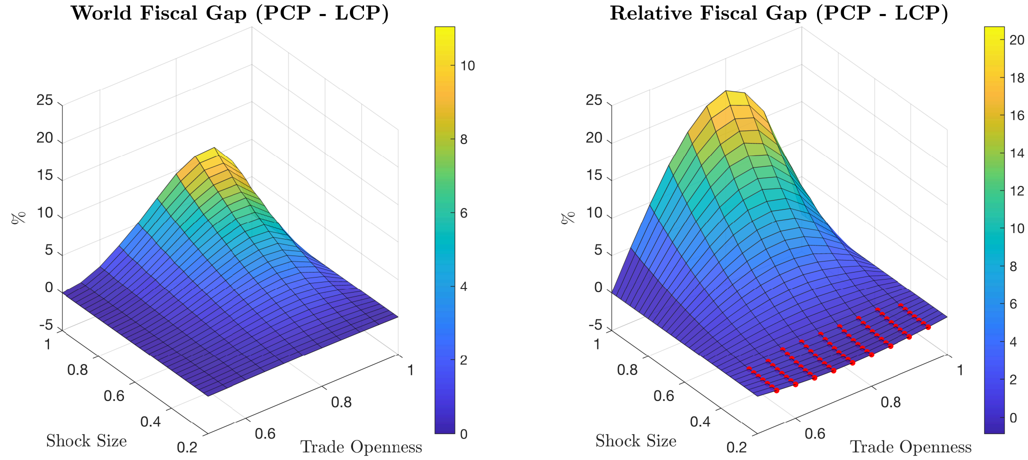

Figure A8: Trade Openness and Optimal Fiscal Policy ($\theta = 1$)

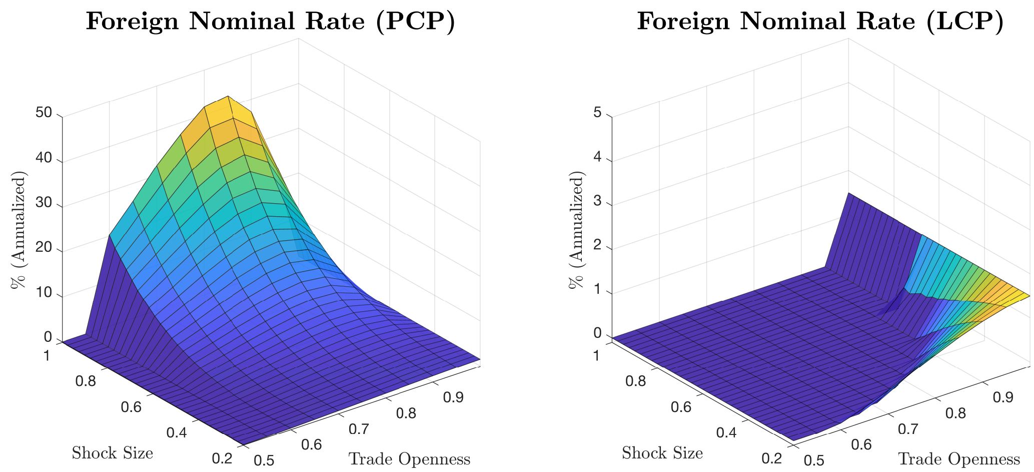

Figure A9: Trade Openness and Optimal Monetary Policy ($\alpha = 0.75$)

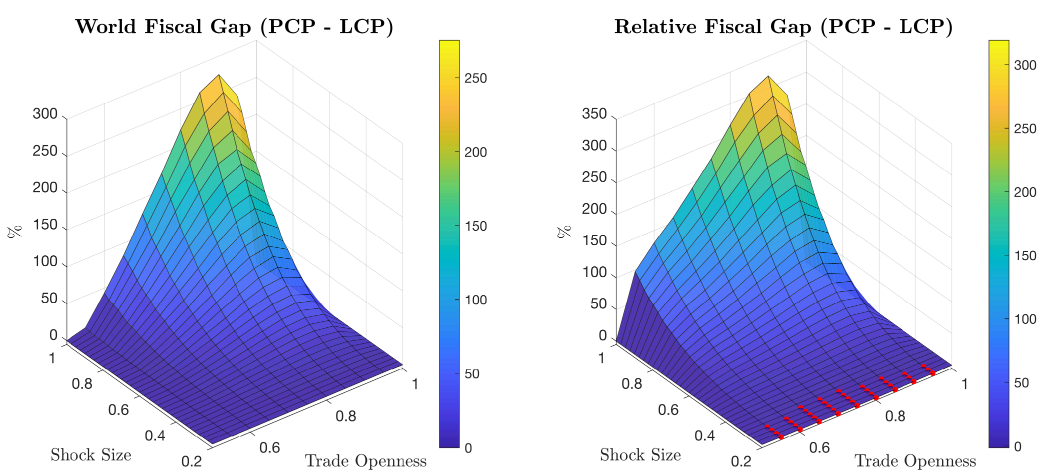

Figure A10: Trade Openness and Optimal Fiscal Policy ($\alpha = 0.75$)

B.1 A Two-Country Model with Various Pricing Regimes

This section presents the log-linearized equilibrium conditions necessary to define equilibrium under each pricing regime. The variables are specified for each country, Home and Foreign. The transformation of these variables into World and Relative terms follows the approach detailed in the appendix of Engel (2011). Since the first-best allocation is identical across the regime, we provide the equilibrium conditions only in Section B.2.1.

B.2 Log-linearized Equilibrium Conditions

B.2.1 Equilibrium conditions under PCP

Given the path of the four policy instruments $\{R_t, R_t^*, G_t, G_t^*\}$, the equilibrium under flexible exchange rates consists of endogenous variables $\{\hat{Y}_{H,t}, \hat{Y}_{F,t}, \hat{\pi}_{H,t}, \hat{\pi}^*_{F,t}, \hat{C}_t, \hat{C}_t^*, \hat{T}_t, \hat{Q}_t\}$ and exogenous variables $\{\hat{Z}_t, \hat{Z}_t^*\}$ that satisfy:

- Aggregate demand:

$$\mathbb{E}_t\hat{C}_{t+1} = \hat{C}_t + \frac{1}{\rho}\left\{\hat{R}_t - \left[\mathbb{E}_t\pi_{H,t+1} + (1-\nu)(\mathbb{E}_t\hat{T}_{t+1} - \hat{T}_t)\right] + \mathbb{E}_t\hat{Z}_{t+1} - \hat{Z}_t\right\}$$
$$\mathbb{E}_t\hat{C}^*_{t+1} = \hat{C}^*_t + \frac{1}{\rho}\left\{\hat{R}^*_t - \left[\mathbb{E}_t\pi^*_{F,t+1} - (1-\nu)(\mathbb{E}_t\hat{T}_{t+1} - \hat{T}_t)\right] + \mathbb{E}_t\hat{Z}^*_{t+1} - \hat{Z}^*_t\right\}$$

- Market clearing:

$$\hat{Y}_{H,t} = c_y\left\{\nu\hat{C}_t + (1-\nu)\hat{C}^*_t + 2\theta\nu(1-\nu)\hat{T}_t\right\} + (1-c_y)\hat{G}_t$$
$$\hat{Y}_{F,t} = c_y\left\{(1-\nu)\hat{C}_t + \nu\hat{C}^*_t - 2\theta\nu(1-\nu)\hat{T}_t\right\} + (1-c_y)\hat{G}^*_t$$

- Aggregate supply:

$$\begin{aligned}\pi_{H,t} = \beta\mathbb{E}_t\pi_{H,t+1} + \kappa_H\bigg\{&(\frac{\rho}{c_y} + \eta)\tilde{Y}_{H,t}\\ &- 2\nu(1-\nu)(\rho\theta-1)\tilde{T}_t + (1-\nu)\hat{f}_t - \rho\frac{1-c_y}{c_y}\tilde{G}_t\bigg\}\\ \pi^*_{F,t} = \beta\mathbb{E}_t\pi^*_{F,t+1} + \kappa_F\bigg\{&(\frac{\rho}{c_y} + \eta)\tilde{Y}_{F,t}\\ &+ 2\nu(1-\nu)(\rho\theta-1)\tilde{T}_t - (1-\nu)\hat{f}_t - \rho\frac{1-c_y}{c_y}\tilde{G}^*_t\bigg\}\end{aligned}$$

- Risk sharing and demand imbalances:

$$\hat{Q}_t = \rho(\hat{C}_t - \hat{C}_t^*) + (\hat{Z}_t^* - \hat{Z}_t)$$
$$\hat{Q}_t = (2\nu - 1)\hat{T}_t$$

- First best allocation

$$(\rho + c_y\eta)\hat{Y}_{H,t}^{fb} = c_y 2\nu(1-\nu)(\rho\theta - 1)\hat{T}_t^{fb} + (1-c_y)\rho\hat{G}_t^{fb} + c_y(\nu\hat{Z}_t + (1-\nu)\hat{Z}_t^*)$$
$$(\rho + c_y\eta)\hat{Y}_{F,t}^{fb} = -c_y 2\nu(1-\nu)(\rho\theta - 1)\hat{T}_t^{fb} + (1-c_y)\rho\hat{G}_t^{*,fb} + c_y((1-\nu)\hat{Z}_t + \nu\hat{Z}_t^*)$$
$$c_y[4\nu(1-\nu)\rho\theta + (2\nu-1)^2]\hat{T}_t^{fb} = \rho(\hat{Y}_{H,t}^{fb} - \hat{Y}_{F,t}^{fb}) - (1-c_y)\rho(\hat{G}_t^{fb} - \hat{G}_t^{*,fb}) - c_y(2\nu-1)(\hat{Z}_t - \hat{Z}_t^*)$$
$$\eta\hat{Y}_{H,t}^{fb} = -\rho\hat{G}_t^{fb}$$
$$\eta\hat{Y}_{F,t}^{fb} = -\rho\hat{G}_t^{*,fb}$$

B.2.2 Equilibrium conditions under LCP

Given the path of the four policy instruments $\{R_t, R_t^*, G_t, G_t^*\}$, the equilibrium under flexible exchange rates consists of endogenous variables $\{\hat{Y}_{H,t}, \hat{Y}_{F,t}, \pi_{H,t}, \pi_{H,t}^*, \pi_{F,t}^*, \pi_{F,t}, \hat{C}_{H,t}, \hat{C}_{F,t}, \hat{T}_t, \hat{T}_t^*, \hat{Q}_t, \hat{m}_t\}$ and exogenous variables $\{\hat{Z}_t, \hat{Z}_t^*\}$ that satisfy:

- Aggregate demand:

$$\mathbb{E}_t\hat{C}_{t+1} = \hat{C}_t + \frac{1}{\rho}\left\{\hat{R}_t - \left[\mathbb{E}_t\pi_{H,t+1} + (1-\nu)(\mathbb{E}_t\hat{T}_{t+1} - \hat{T}_t)\right] + \mathbb{E}_t\hat{Z}_{t+1} - \hat{Z}_t\right\}$$
$$\mathbb{E}_t\hat{C}_{t+1}^* = \hat{C}_t^* + \frac{1}{\rho}\left\{\hat{R}_t^* - \left[\mathbb{E}_t\pi_{F,t+1}^* + (1-\nu)(\mathbb{E}_t\hat{T}_{t+1}^* - \hat{T}_t^*)\right] + \mathbb{E}_t\hat{Z}_{F,t+1} - \hat{Z}_t^*\right\}$$

- Market clearing:

$$\hat{Y}_{H,t} = c_y\left\{\nu\hat{C}_t + (1-\nu)\hat{C}_t^* + \theta\nu(1-\nu)(\hat{T}_t - \hat{T}_t^*)\right\} + (1-c_y)\hat{G}_t$$
$$\hat{Y}_{F,t} = c_y\left\{(1-\nu)\hat{C}_t + \nu\hat{C}_t^* - \theta\nu(1-\nu)(\hat{T}_t - \hat{T}_t^*)\right\} + (1-c_y)\hat{G}_t^*$$

- Aggregate supply:

$$\pi_{H,t} = \beta\mathbb{E}_t\pi_{H,t+1} + \kappa_H \left\{ (\frac{\rho}{c_y} + \eta)\tilde{Y}_{H,t} \right.$$
$$\left. - \nu(1-\nu)(\rho\theta - 1)(\tilde{T}_t - \tilde{T}_t^*) + \frac{1-\nu}{2}(\hat{T}_t + \hat{T}_t^*) + (1-\nu)\hat{m}_t - \rho\frac{1-c_y}{c_y}\tilde{G}_t \right\}$$
$$\pi_{H,t}^* = \beta\mathbb{E}_t\pi_{H,t+1}^* + \kappa_H \left\{ (\frac{\rho}{c_y} + \eta)\tilde{Y}_{H,t} \right.$$
$$\left. - \nu(1-\nu)(\rho\theta - 1)(\tilde{T}_t - \tilde{T}_t^*) - \frac{\nu}{2}(\hat{T}_t + \hat{T}_t^*) - \nu\hat{m}_t - \rho\frac{1-c_y}{c_y}\tilde{G}_t \right\}$$
$$\pi_{F,t}^* = \beta\mathbb{E}_t\pi_{F,t+1}^* + \kappa_F \left\{ (\frac{\rho}{c_y} + \eta)\tilde{Y}_{F,t} \right.$$
$$\left. + \nu(1-\nu)(\rho\theta - 1)(\tilde{T}_t - \tilde{T}_t^*) + \frac{1-\nu}{2}(\hat{T}_t + \hat{T}_t^*) - (1-\nu)\hat{m}_t - \rho\frac{1-c_y}{c_y}\tilde{G}_t^* \right\}$$
$$\pi_{F,t} = \beta\mathbb{E}_t\pi_{F,t+1} + \kappa_F \left\{ (\frac{\rho}{c_y} + \eta)\tilde{Y}_{F,t} \right.$$
$$\left. + \nu(1-\nu)(\rho\theta - 1)(\tilde{T}_t - \tilde{T}_t^*) - \frac{\nu}{2}(\hat{T}_t + \hat{T}_t^*) + \nu\hat{m}_t - \rho\frac{1-c_y}{c_y}\tilde{G}_t^* \right\}$$

- Risk sharing:

$$\hat{Q}_t = \rho(\hat{C}_t - \hat{C}_t^*) + (\hat{Z}_t^* - \hat{Z}_t)$$
$$\hat{Q}_t = \hat{m}_t + \frac{2\nu - 1}{2}(\hat{T}_t - \hat{T}_t^*)$$

- Dynamic equations for relative price:

$$\hat{T}_t - \hat{T}_{t-1} = \pi_{F,t} - \pi_{H,t}$$
$$\hat{T}_t^* - \hat{T}_{t-1}^* = \pi_{H,t}^* - \pi_{F,t}^*$$

B.3 Welfare Loss Functions

B.3.1 Welfare Loss Function (PCP)

$$2\mathcal{L}_t^W = -\left(\frac{\rho}{c_y} + \eta\right)\left(\tilde{Y}_t^W\right)^2 - \left(\frac{\rho}{c_y D} + \eta\right)\left(\tilde{Y}_t^R\right)^2 - \rho\frac{1-c_y}{c_y}\left(\tilde{G}_t^W\right)^2 - \rho(1-c_y)\left(\frac{1-c_y}{c_y D} + 1\right)\left(\tilde{G}_t^R\right)^2$$
$$+ 2\rho\frac{1-c_y}{c_y}\tilde{Y}_t^W\tilde{G}_t^W + 2\rho\frac{1-c_y}{c_y D}\tilde{Y}_t^R\tilde{G}_t^R - \frac{\sigma}{2}\left(\frac{1}{\kappa_H}\pi_{H,t}^2 + \frac{1}{\kappa_F}{\pi^*{}_{F,t}}^2\right)$$
where $D \equiv (2\nu - 1)^2 + 4\rho\theta\nu(1-\nu)$

B.3.2 Welfare Loss Function (LCP)

$$
\begin{aligned}
2\mathcal{L}_t^W = & -\left(\frac{\rho}{c_y}+\eta\right)\left(\tilde{Y}_t^W\right)^2 - \left(\frac{\rho}{c_y D}+\eta\right)\left(\tilde{Y}_t^R\right)^2 - \rho\frac{1-c_y}{c_y}\left(\tilde{G}_t^W\right)^2 - \rho(1-c_y)\left(\frac{1-c_y}{c_y D}+1\right)\left(\tilde{G}_t^R\right)^2 \\
& + 2\rho\frac{1-c_y}{c_y}\tilde{Y}_t^W\tilde{G}_t^W + 2\rho\frac{1-c_y}{c_y D}\tilde{Y}_t^R\tilde{G}_t^R - \frac{c_y\theta\nu(1-\nu)}{D}\left(\hat{m}_t\right)^2 - c_y\theta\nu(1-\nu)\left(\hat{T}_t^W\right)^2 \\
& - \frac{c_y\sigma}{2}\left(\frac{1-(1-\nu)c_y}{c_y\kappa_H}\pi_{H,t}^2 + \frac{1-\nu}{\kappa_H}(\pi_{H,t}^*)^2 + \frac{1-(1-\nu)c_y}{c_y\kappa_F}(\pi_{F,t}^*)^2 + \frac{1-\nu}{\kappa_F}\pi_{F,t}^2\right)
\end{aligned}
$$

B.4 First Best Allocation

The first-best allocation is the equilibrium in which prices are fully flexible, capital markets are complete, and the steady state markups are always neutralized with an appropriate subsidy. Since this efficient allocation is invariant to the pricing regime, it provides a useful benchmark for comparing the welfare implications across the three regimes. Any variable marked with a hat and the subscript fb (e.g., $\hat{X}_t^{fb}$) represents its log–linearized version under the first–best allocation.

The log-linearized risk-sharing condition under perfect risk-sharing is:

$$\hat{Q}_t^{fb} = \rho\left(\hat{C}_t^{fb} - \hat{C}_t^{*,fb}\right) - \left(\hat{Z}_t - \hat{Z}_t^*\right). \tag{B1}$$

Furthermore, log-linearizing the real exchange rate yields:

$$\hat{Q}_t^{fb} = (2\nu-1)\hat{T}_t^{fb}. \tag{B2}$$

Combining the preceding results with the market–clearing conditions yields:

$$\frac{\rho}{c_y}\hat{Y}_{H,t}^{fb} = \rho\hat{C}_t^{fb} + (1-\nu)[2\nu(\rho\theta-1)+1]\hat{T}_t^{fb} + \rho\frac{1-c_y}{c_y}\hat{G}_t^{fb} - (1-\nu)\left(\hat{Z}_t - \hat{Z}_t^*\right). \tag{B3}$$

Log-linearizing the Home optimal pricing equation under flexible prices yields:

$$\eta\hat{Y}_{H,t}^{fb} = -(1-\nu)\hat{T}_t^{fb} - \rho\hat{C}_t^{fb} + \hat{Z}_t. \tag{B4}$$

Eliminating $\hat{C}_t^{fb}$ by combining the equations (B3) and (B4) results in:

$$\left(\frac{\rho}{c_y}+\eta\right)\hat{Y}_{H,t}^{fb} = 2\nu(1-\nu)(\rho\theta-1)\hat{T}_t^{fb} + \rho\frac{1-c_y}{c_y}\hat{G}_t^{fb} - (1-\nu)\left(\hat{Z}_t - \hat{Z}_t^*\right) + \hat{Z}_t. \tag{B5}$$

An analogous derivation for Foreign yields:

$$\left(\frac{\rho}{c_y}+\eta\right)\hat{Y}_{F,t}^{fb} = -2\nu(1-\nu)(\rho\theta-1)\hat{T}_t^{fb} + \rho\frac{1-c_y}{c_y}\hat{G}_t^{*,fb} + (1-\nu)\left(\hat{Z}_t - \hat{Z}_t^*\right) + \hat{Z}_t^*. \tag{B6}$$

Adding the equations (B5) and (B6) yields:

$$\left(\frac{\rho}{c_y}+\eta\right)\hat{Y}_t^{W,fb} = \rho\frac{1-c_y}{c_y}\hat{G}_t^{W,fb} + \hat{Z}_t^W. \tag{B7}$$

Subtracting the equation (B6) from (B5) yields:

$$\left(\frac{\rho}{c_y}+\eta\right)\hat{Y}_t^{R,fb}=2\nu(1-\nu)(\rho\theta-1)\hat{T}_t^{fb}+\rho\frac{1-c_y}{c_y}\hat{G}_t^{R,fb}+(2\nu-1)\hat{Z}_t^{R}. \tag{B8}$$

The efficient terms of trade is given by:

$$D\hat{T}_t^{fb}=\frac{2\rho}{c_y}\hat{Y}_t^{R,fb}-2\rho\frac{1-c_y}{c_y}\hat{G}_t^{R,fb}-2(2\nu-1)\hat{Z}_t^{R}, \tag{B9}$$

which is obtained by combining the two log-linearized aggregate market-clearing conditions with the risk-sharing condition (B1) and Equation (B2).

The first-best government spending follows:

$$\begin{aligned} & V'(N_t^{fb})=J'(G_t^{fb}). \\ \Leftrightarrow\quad & \eta\hat{Y}_{H,t}^{fb}=-\rho\hat{G}_t^{fb} \end{aligned} \tag{B10}$$

Finally, substituting (B8), (B9), and (B10) yields the relationship:

$$\hat{T}_t^{fb}=-2\eta\hat{Y}_t^{R} \tag{B11}$$

B.5 Derivation of the Quadratic Loss Function

In this section, we derive the quadratic welfare loss functions for each pricing regime, following closely the approach in Engel (2011). For all regimes, the aggregate period utility for Home and Foreign is given by:

$$\begin{aligned} & U(C_t)-V(N_t)+J(G_t)+U(C_t^*)-V(N_t^*)+J(G_t^*) \\ & =\left(Z_t\frac{C_t^{1-\rho}}{1-\rho}-\frac{N_t^{1+\eta}}{1+\eta}+\Psi\frac{G_t^{1-\rho}}{1-\rho}\right)+\left(Z_t^*\frac{(C_t^*)^{1-\rho}}{1-\rho}-\frac{(N_t^*)^{1+\eta}}{1+\eta}+\Psi\frac{(G_t^*)^{1-\rho}}{1-\rho}\right). \end{aligned} \tag{B12}$$

We assume an efficient steady state so that the subsidy satisfies $\frac{(\sigma-1)(1-\tau)}{\sigma}=1$. Moreover, in the efficient steady-state, $U'(C^{fb})=V'(N^{fb})=J'(G^{fb})$ and $U'(C^{*,fb})=V'(N^{*,fb})=J'(G^{*,fb})$ holds. The second-order approximation of the world utility (B12) yields:

$$\begin{aligned} 2\mathcal{L}_t =& \hat{C}_t+\hat{C}_t^*-\frac{1}{c_y}\hat{Y}_{H,t}-\frac{1}{c_y}\hat{Y}_{F,t}+\frac{1-c_y}{c_y}\hat{G}_t+\frac{1-c_y}{c_y}\hat{G}_t^*+\hat{Z}_t\hat{C}_t+\hat{Z}_t^*\hat{C}_t^* \\ & +\frac{1-\rho}{2}\hat{C}_t^2+\frac{1-\rho}{2}(\hat{C}_t^*)^2-\frac{1}{c_y}\frac{1+\eta}{2}\hat{Y}_{H,t}^2-\frac{1}{c_y}\frac{1+\eta}{2}\hat{Y}_{F,t}^2+\frac{1-c_y}{c_y}\frac{1-\rho}{2}\hat{G}_t^2+\frac{1-c_y}{c_y}\frac{1-\rho}{2}(\hat{G}_t^*)^2+t.i.p. \end{aligned} \tag{B13}$$

Subtracting the first–best welfare from (B13) and regrouping terms into the world and relative components gives:

$$
\begin{aligned}
2\left(\mathcal{L}_t - \mathcal{L}_t^{fb}\right) =& \tilde{C}_t + \tilde{C}_t^* - \frac{1}{c_y}\tilde{Y}_{H,t} - \frac{1}{c_y}\tilde{Y}_{F,t} + \frac{1-c_y}{c_y}\tilde{G}_t + \frac{1-c_y}{c_y}\tilde{G}_t^* + \hat{Z}_t\tilde{C}_t + \hat{Z}_t^*\tilde{C}_t^* \\
&+ \frac{1-\rho}{2}\tilde{C}_t^2 + \frac{1-\rho}{2}(\tilde{C}_t^*)^2 - \frac{1}{c_y}\frac{1+\eta}{2}\tilde{Y}_{H,t}^2 - \frac{1}{c_y}\frac{1+\eta}{2}\tilde{Y}_{F,t}^2 \\
&+ \frac{1-c_y}{c_y}\frac{1-\rho}{2}\tilde{G}_t^2 + \frac{1-c_y}{c_y}\frac{1-\rho}{2}(\tilde{G}_t^*)^2 \\
&+ (1-\rho)\left(\hat{C}_t^{fb}\tilde{C}_t + \hat{C}_t^{*,fb}\tilde{C}_t^*\right) - \frac{1}{c_y}(1+\eta)\left(\hat{Y}_{H,t}^{fb}\tilde{Y}_{H,t} + \hat{Y}_{F,t}^{fb}\tilde{Y}_{F,t}\right) \\
&+ \frac{1-c_y}{c_y}(1-\rho)\left(\hat{G}_t^{fb}\tilde{G}_t + \hat{G}_t^{*,fb}\tilde{G}_t^*\right) + t.i.p.
\end{aligned}
$$

$$
\begin{aligned}
=& 2\tilde{C}_t^W - 2\frac{1}{c_y}\tilde{Y}_t^W + 2\frac{1-c_y}{c_y}\tilde{G}_t^W + 2\hat{Z}_t^W\tilde{C}_t^W + 2\hat{Z}_t^R\tilde{C}_t^R \\
&+ (1-\rho)\left[\left(\tilde{C}_t^W\right)^2 + \left(\tilde{C}_t^R\right)^2\right] - \frac{1}{c_y}(1+\eta)\left[\left(\tilde{Y}_t^W\right)^2 + \left(\tilde{Y}_t^R\right)^2\right] \\
&+ \frac{1-c_y}{c_y}(1-\rho)\left[\left(\tilde{G}_t^W\right)^2 + \left(\tilde{G}_t^R\right)^2\right] \\
&+ 2(1-\rho)\left(\hat{C}_t^{W,fb}\tilde{C}_t^W + \hat{C}_t^{R,fb}\tilde{C}_t^R\right) - \frac{2}{c_y}(1+\eta)\left(\hat{Y}_t^{W,fb}\tilde{Y}_t^W + \hat{Y}_t^{R,fb}\tilde{Y}_t^R\right) \\
&+ 2\frac{1-c_y}{c_y}(1-\rho)\left(\hat{G}_t^{W,fb}\tilde{G}_t^W + \hat{G}_t^{R,fb}\tilde{G}_t^R\right) + t.i.p. \qquad \text{(B14)}
\end{aligned}
$$

The subsequent steps depend on the pricing regime; accordingly, we develop the quadratic loss function separately for each regime in the following subsections. Section B.5.1 and B.5.2 describe the steps to derive the welfare cost function under PCP. Section B.5.3 and B.5.4 illustrate the steps under LCP. For each regime, we first state the relevant first– and second–order conditions and then develop the equation (B14) for that regime.

B.5.1 Useful First- and Second-Order Conditions (PCP)

By rearranging the country-specific resource constraints, risk-sharing condition, and the definition of the real exchange rate ($\hat{Q}_t = (2\nu - 1)\hat{T}_t$), we obtain:

$$
\hat{Y}_{H,t} = c_y\left[\hat{C}_t + \frac{D-(2\nu-1)}{2\rho}\hat{T}_t - \frac{2(1-\nu)}{\rho}\hat{Z}_t^R\right] + (1-c_y)\hat{G}_t, \qquad \text{(B15)}
$$

$$
\hat{Y}_{H,t} = c_y\left[\hat{C}_t^* + \frac{D+(2\nu-1)}{2\rho}\hat{T}_t + \frac{2\nu}{\rho}\hat{Z}_t^R\right] + (1-c_y)\hat{G}_t, \qquad \text{(B16)}
$$

$$
\hat{Y}_{F,t} = c_y\left[\hat{C}_t^* - \frac{D-(2\nu-1)}{2\rho}\hat{T}_t + \frac{2(1-\nu)}{\rho}\hat{Z}_t^R\right] + (1-c_y)\hat{G}_t^*. \qquad \text{(B17)}
$$

Then, by combining the equations (B15), (B16), and (B17), we can express the world consumption gap, relative consumption gap, and the relative price gap in terms of the output, government spending, and the shock as follow:

$$\hat{C}_t^W = \frac{1}{c_y}\hat{Y}_t^W - \frac{1-c_y}{c_y}\hat{G}_t^W, \tag{B18}$$

$$\hat{C}_t^R = \frac{1}{c_y}\frac{2\nu-1}{D}\hat{Y}_t^R - \frac{1-c_y}{c_y}\frac{2\nu-1}{D}\hat{G}_t^R + \frac{D-(2\nu-1)^2}{\rho D}\hat{Z}_t^R, \tag{B19}$$

$$\hat{T}_t = \frac{1}{c_y}\frac{2\rho}{D}\hat{Y}_t^R - \frac{1-c_y}{c_y}\frac{2\rho}{D}\hat{G}_t^R - \frac{2(2\nu-1)}{D}\hat{Z}_t^R. \tag{B20}$$

Adding (B15) to (B17) yields (B18), subtracting (B17) from (B16) yields (B20), and substituting (B20) into the risk-sharing condition yields (B19).

Similarly, the corresponding equations under efficient equilibrium are:

$$\hat{C}_t^{W,fb} = \frac{1}{c_y}\hat{Y}_t^{W,fb} - \frac{1-c_y}{c_y}\hat{G}_t^{W,fb}, \tag{B21}$$

$$\hat{C}_t^{R,fb} = \frac{1}{c_y}\frac{2\nu-1}{D}\hat{Y}_t^{R,fb} - \frac{1-c_y}{c_y}\frac{2\nu-1}{D}\hat{G}_t^{R,fb} + \frac{D-(2\nu-1)^2}{\rho D}\hat{Z}_t^{R,fb}, \tag{B22}$$

$$\hat{T}_t^{fb} = \frac{1}{c_y}\frac{2\rho}{D}\hat{Y}_t^{R,fb} - \frac{1-c_y}{c_y}\frac{2\rho}{D}\hat{G}_t^{R,fb} - \frac{2(2\nu-1)}{D}\hat{Z}_t^R. \tag{B23}$$

One can express these equations as the efficient gap terms by subtracting the corresponding sticky-price equilibrium and the efficient equilibrium equations:

$$\tilde{C}_t^W = \frac{1}{c_y}\tilde{Y}_t^W - \frac{1-c_y}{c_y}\tilde{G}_t^W, \tag{B24}$$

$$\tilde{C}_t^R = \frac{1}{c_y}\frac{2\nu-1}{D}\tilde{Y}_t^R - \frac{1-c_y}{c_y}\frac{2\nu-1}{D}\tilde{G}_t^R, \tag{B25}$$

$$\tilde{T}_t = \frac{1}{c_y}\frac{2\rho}{D}\tilde{Y}_t^R - \frac{1-c_y}{c_y}\frac{2\rho}{D}\tilde{G}_t^R. \tag{B26}$$

The resource constraint for each country can written as follows:

$$Y_{H,t} = \Delta_{H,t}\left\{\left(\frac{P_{H,t}}{P_t}\right)^{-\theta}\left[\nu C_t + (1-\nu)Q^\theta C_t^*\right] + G_t\right\},$$

$$Y_{F,t} = \Delta_{F,t}^*\left\{\left(\frac{P_{F,t}^*}{P_t^*}\right)^{-\theta}\left[(1-\nu)C_t + \nu Q^\theta C_t^*\right] + G_t^*\right\},$$

where $\Delta_{H,t} \equiv \int_0^1 \left(\frac{P_{H,t}(i)}{P_{H,t}}\right)^{-\sigma} di$ and $\Delta_{F,t}^* \equiv \int_0^1 \left(\frac{P_{F,t}^*(i)}{P_{F,t}^*}\right)^{-\sigma} di$ are the price dispersions for the Home and the Foreign goods, respectively. By approximating the resource constraints in a second order and taking a summation of them, we derive the following[3]:

[3]The inflation rates appear because of the second-order approximation of price dispersion terms. Please refer to Woodford (2003) chapter 6 for more detail.

$$
\begin{aligned}
2\hat{C}_t^W - \frac{2}{c_y}\hat{Y}_t^W + \frac{2(1-c_y)}{c_y}\hat{G}_t^W = & - \left[\left(\hat{C}_t^W\right)^2 + \left(\hat{C}_t^R\right)^2\right] + \frac{1}{c_y}\left[\left(\hat{Y}_t^W\right)^2 + \left(\hat{Y}_t^R\right)^2\right] \\
& - \frac{1-c_y}{c_y}\left[\left(\hat{G}_t^W\right)^2 + \left(\hat{G}_t^R\right)^2\right] - \theta\nu(1-\nu)\hat{T}_t^2 \\
& - \frac{1}{c_y}\frac{\sigma}{2\kappa}\left[\hat{\pi}_{H,t}^2 + \left(\hat{\pi}_{F,t}^*\right)^2\right]
\end{aligned} \tag{B27}
$$

Then, by subtracting the corresponding efficient equilibrium equation from the equation (B27), we get:

$$
\begin{aligned}
2\tilde{C}_t^W - \frac{2}{c_y}\tilde{Y}_t^W + \frac{2(1-c_y)}{c_y}\tilde{G}_t^W = & - \left[\left(\tilde{C}_t^W\right)^2 + \left(\tilde{C}_t^R\right)^2\right] + \frac{1}{c_y}\left[\left(\tilde{Y}_t^W\right)^2 + \left(\tilde{Y}_t^R\right)^2\right] \\
& - \frac{1-c_y}{c_y}\left[\left(\tilde{G}_t^W\right)^2 + \left(\tilde{G}_t^R\right)^2\right] - \theta\nu(1-\nu)\tilde{T}_t^2 \\
& - \left(\hat{C}_t^{fb,W}\tilde{C}_t^W + \hat{C}_t^{fb,R}\tilde{C}_t^R\right) + \frac{2}{c_y}\left(\hat{Y}_t^{fb,W}\tilde{Y}_t^W + \hat{Y}_t^{fb,R}\tilde{Y}_t^R\right) \\
& - \frac{2(1-c_y)}{c_y}\left(\hat{G}_t^{fb,W}\tilde{G}_t^W + \hat{G}_t^{fb,R}\tilde{G}_t^R\right) - 2\theta\nu(1-\nu)\hat{T}^{fb}\tilde{T}_t \\
& - \frac{1}{c_y}\frac{\sigma}{2\kappa}\left[\hat{\pi}_{H,t}^2 + \left(\hat{\pi}_{F,t}^*\right)^2\right].
\end{aligned} \tag{B28}
$$

Equations (B21), (B22), (B23), (B24), (B25), (B26), and (B28) will be used in the next subsection.

B.5.2 Further Manipulation of the Welfare Loss Function (PCP)

We use the second-order approximations of the resource constraints to fully approximate the welfare loss function into the second-order terms. By substituting the Equation (B28) into (B14), we obtain:

$$
\begin{aligned}
2\left(\mathcal{L}_t - \mathcal{L}_t^{fb}\right) = & - 2\rho\hat{C}_t^{R,fb}\tilde{C}_t^R + 2\hat{Z}_t^R\tilde{C}_t^R - 2\theta\nu(1-\nu)\hat{T}^{fb}\tilde{T}_t \\
& - \frac{2}{c_y}\eta\hat{Y}_t^{R,fb}\tilde{Y}_t^R - 2\frac{1-c_y}{c_y}\rho\hat{G}_t^{R,fb}\tilde{G}_t^R \\
& - 2\rho\hat{C}_t^{W,fb}\tilde{C}_t^W + 2\hat{Z}_t^W\tilde{C}_t^W \\
& - \frac{2}{c_y}\eta\hat{Y}_t^{W,fb}\tilde{Y}_t^W - 2\frac{1-c_y}{c_y}\rho\hat{G}_t^{W,fb}\tilde{G}_t^W \\
& - \rho\left[\left(\tilde{C}_t^W\right)^2 + \left(\tilde{C}_t^R\right)^2\right] - \frac{1}{c_y}\eta\left[\left(\tilde{Y}_t^W\right)^2 + \left(\tilde{Y}_t^R\right)^2\right] \\
& - \frac{1-c_y}{c_y}\rho\left[\left(\tilde{G}_t^W\right)^2 + \left(\tilde{G}_t^R\right)^2\right] - \theta\nu(1-\nu)\tilde{T}_t^2 \\
& - \frac{1}{c_y}\frac{\sigma}{2\kappa}\left[\hat{\pi}_{H,t}^2 + \left(\hat{\pi}_{F,t}^*\right)^2\right] + t.i.p.
\end{aligned} \tag{B29}
$$

To recast the welfare loss in terms of output gaps, fiscal gaps, and inflation, we eliminate the interaction terms between the relative flexible variables ($\hat{X}_t^{R,fb}$) and the relative gap variables

($\tilde{X}_t^R$). By using the relationship $\tilde{C}_t^R = \frac{2\nu-1}{2\rho}\tilde{T}_t$[4] and the equation (B10), we can rearrange the first two rows of the equation (B29) as follows:

$$\begin{aligned}
2\left(\mathcal{L}_t - \mathcal{L}_t^{fb}\right) = &-\left[(2\nu-1)\hat{C}_t^{R,fb} + 2\theta\nu(1-\nu)\hat{T}_t^{fb} - \frac{2\nu-1}{\rho}\hat{Z}_t^R\right]\tilde{T}_t \\
&- 2\eta\hat{Y}_t^{R,fb}\left(\frac{1}{c_y}\tilde{Y}_t^R - \frac{1-c_y}{c_y}\tilde{G}_t^R\right) \\
&- 2\rho\hat{C}_t^{W,fb}\tilde{C}_t^W + 2\hat{Z}_t^W\tilde{C}_t^W \\
&- \frac{2}{c_y}\eta\hat{Y}_t^{W,fb}\tilde{Y}_t^W - 2\frac{1-c_y}{c_y}\rho\hat{G}_t^{W,fb}\tilde{G}_t^W \\
&- \rho\left[\left(\tilde{C}_t^W\right)^2 + \left(\tilde{C}_t^R\right)^2\right] - \frac{1}{c_y}\eta\left[\left(\tilde{Y}_t^W\right)^2 + \left(\tilde{Y}_t^R\right)^2\right] \\
&- \frac{1-c_y}{c_y}\rho\left[\left(\tilde{G}_t^W\right)^2 + \left(\tilde{G}_t^R\right)^2\right] - \theta\nu(1-\nu)\tilde{T}_t^2 \\
&- \frac{1}{c_y}\frac{\sigma}{2\kappa}\left[\hat{\pi}_{H,t}^2 + \left(\hat{\pi}_{F,t}^*\right)^2\right] + t.i.p.
\end{aligned}$$

Then, substituting the equations (B22) and (B23) into the first row, and the equation (B26) into the second row yields:

$$\begin{aligned}
2\left(\mathcal{L}_t - \mathcal{L}_t^{fb}\right) = &-\left(\frac{1}{c_y}\hat{Y}_t^{R,fb} - \frac{1-c_y}{c_y}\hat{G}_t^{R,fb} - \frac{2\nu-1}{\rho}\hat{Z}_t^R\right)\tilde{T}_t \\
&- \frac{\eta}{\rho}D\hat{Y}_t^{R,fb}\tilde{T}_t \\
&- 2\rho\hat{C}_t^{W,fb}\tilde{C}_t^W + 2\hat{Z}_t^W\tilde{C}_t^W \\
&- \frac{2}{c_y}\eta\hat{Y}_t^{W,fb}\tilde{Y}_t^W - 2\frac{1-c_y}{c_y}\rho\hat{G}_t^{W,fb}\tilde{G}_t^W \\
&- \rho\left[\left(\tilde{C}_t^W\right)^2 + \left(\tilde{C}_t^R\right)^2\right] - \frac{1}{c_y}\eta\left[\left(\tilde{Y}_t^W\right)^2 + \left(\tilde{Y}_t^R\right)^2\right] \\
&- \frac{1-c_y}{c_y}\rho\left[\left(\tilde{G}_t^W\right)^2 + \left(\tilde{G}_t^R\right)^2\right] - \theta\nu(1-\nu)\tilde{T}_t^2 \\
&- \frac{1}{c_y}\frac{\sigma}{2\kappa}\left[\hat{\pi}_{H,t}^2 + \left(\hat{\pi}_{F,t}^*\right)^2\right] + t.i.p.
\end{aligned}$$

[4] This relationship can be easily derived from the equations (B25) and (B26).

Using the equation (B23) and the relation between the flexible relative price and the flexible relative output gap (B11) rids out the first two rows.

$$
\begin{aligned}
2\left(\mathcal{L}_t - \mathcal{L}_t^{fb}\right) =& -2\rho \hat{C}_t^{W,fb}\tilde{C}_t^W + 2\hat{Z}_t^W\tilde{C}_t^W \\
& - \frac{2}{c_y}\eta\hat{Y}_t^{W,fb}\tilde{Y}_t^W - 2\frac{1-c_y}{c_y}\rho\hat{G}_t^{W,fb}\tilde{G}_t^W \\
& - \rho\left[\left(\tilde{C}_t^W\right)^2 + \left(\tilde{C}_t^R\right)^2\right] - \frac{1}{c_y}\eta\left[\left(\tilde{Y}_t^W\right)^2 + \left(\tilde{Y}_t^R\right)^2\right] \\
& - \frac{1-c_y}{c_y}\rho\left[\left(\tilde{G}_t^W\right)^2 + \left(\tilde{G}_t^R\right)^2\right] - \theta\nu(1-\nu)\tilde{T}_t^2 \\
& - \frac{1}{c_y}\frac{\sigma}{2\kappa}\left[\hat{\pi}_{H,t}^2 + \left(\hat{\pi}_{F,t}^*\right)^2\right] + t.i.p.
\end{aligned}
$$

To substitute out the interaction terms between the world first-best variables ($\hat{X}_t^{W,fb}$) and the world gap variables ($\tilde{X}_t^W$), we first use the equation (B7), (B21) and (B24) to yield:

$$
\begin{aligned}
2\left(\mathcal{L}_t - \mathcal{L}_t^{fb}\right) =& 2\eta\hat{Y}_t^{W,fb}\left(\frac{1}{c_y}\tilde{Y}_t^W - \frac{1-c_y}{c_y}\tilde{G}_t^W\right) \\
& - \frac{2}{c_y}\eta\hat{Y}_t^{W,fb}\tilde{Y}_t^W - 2\frac{1-c_y}{c_y}\rho\hat{G}_t^{W,fb}\tilde{G}_t^W \\
& - \rho\left[\left(\tilde{C}_t^W\right)^2 + \left(\tilde{C}_t^R\right)^2\right] - \frac{1}{c_y}\eta\left[\left(\tilde{Y}_t^W\right)^2 + \left(\tilde{Y}_t^R\right)^2\right] \\
& - \frac{1-c_y}{c_y}\rho\left[\left(\tilde{G}_t^W\right)^2 + \left(\tilde{G}_t^R\right)^2\right] - \theta\nu(1-\nu)\tilde{T}_t^2 \\
& - \frac{1}{c_y}\frac{\sigma}{2\kappa}\left[\hat{\pi}_{H,t}^2 + \left(\hat{\pi}_{F,t}^*\right)^2\right] + t.i.p.
\end{aligned}
$$

Then, using the equation (B10) will substitute out the interaction terms:

$$
\begin{aligned}
2\left(\mathcal{L}_t - \mathcal{L}_t^{fb}\right) =& -\rho\left[\left(\tilde{C}_t^W\right)^2 + \left(\tilde{C}_t^R\right)^2\right] - \frac{1}{c_y}\eta\left[\left(\tilde{Y}_t^W\right)^2 + \left(\tilde{Y}_t^R\right)^2\right] \\
& - \frac{1-c_y}{c_y}\rho\left[\left(\tilde{G}_t^W\right)^2 + \left(\tilde{G}_t^R\right)^2\right] - \theta\nu(1-\nu)\tilde{T}_t^2 \\
& - \frac{1}{c_y}\frac{\sigma}{2\kappa}\left[\hat{\pi}_{H,t}^2 + \left(\hat{\pi}_{F,t}^*\right)^2\right] + t.i.p.
\end{aligned}
$$

Lastly, substituting $\tilde{C}_t^W$, $\tilde{C}_t^R$, and $\tilde{T}_t$ using the equations (B24), (B25), and (B26), and some rearrangement, we can derive the welfare loss function as follows[5]:

$$
\begin{aligned}
2\left(\mathcal{L}_t - \mathcal{L}_t^{fb}\right) = &-\left(\frac{\rho}{c_y}+\eta\right)\left(\tilde{Y}_t^W\right)^2 - \left(\frac{\rho}{c_y D}+\eta\right)\left(\tilde{Y}_t^R\right)^2 \\
&-\rho\frac{1-c_y}{c_y}\left(\tilde{G}_t^W\right)^2 - \rho(1-c_y)\left(\frac{1-c_y}{c_y D}+1\right)\left(\tilde{G}_t^R\right)^2 \\
&+2\rho\frac{1-c_y}{c_y}\tilde{Y}_t^W\tilde{G}_t^W + 2\rho\frac{1-c_y}{c_y D}\tilde{Y}_t^R\tilde{G}_t^R \\
&-\frac{\sigma}{\kappa}\left[(\hat{\pi}_t^W)^2 + (\hat{\pi}_{ppi,t}^R)^2\right]. \qquad \text{(B30)}
\end{aligned}
$$

B.5.3 Useful First- and Second-Order Conditions (LCP)

By rearranging the country-specific resource constraints, risk-sharing condition, and the definition of the real exchange rate ($\hat{Q}_t = (2\nu-1)\hat{T}_t + \hat{m}_t$), we obtain:

$$
\hat{Y}_{H,t} = c_y\left\{\hat{C}_t + \frac{D-(2\nu-1)}{2\rho}\hat{T}_t - \frac{1-\nu}{\rho}\hat{m}_t - \frac{2(1-\nu)}{\rho}\hat{Z}_t^R\right\} + (1-c_y)\hat{G}_t, \qquad \text{(B31)}
$$

$$
\hat{Y}_{H,t} = c_y\left\{\hat{C}_t^* + \frac{D+(2\nu-1)}{2\rho}\hat{T}_t + \frac{1-\nu}{\rho}\hat{m}_t + \frac{2\nu}{\rho}\hat{Z}_t^R\right\} + (1-c_y)\hat{G}_t, \qquad \text{(B32)}
$$

$$
\hat{Y}_{F,t} = c_y\left\{\hat{C}_t^* - \frac{D-(2\nu-1)}{2\rho}\hat{T}_t + \frac{\nu}{\rho}\hat{m}_t + \frac{2(1-\nu)}{\rho}\hat{Z}_t^R\right\} + (1-c_y)\hat{G}_t^*. \qquad \text{(B33)}
$$

Then, by combining the equations (B31), (B32), and (B33), we can express the world consumption gap, relative consumption gap, and the relative price gap in terms of the output, government spending, currency misalignment, and the shock as follow:

$$
\hat{C}_t^W = \frac{1}{c_y}\hat{Y}_t^W - \frac{1-c_y}{c_y}\hat{G}_t^W, \qquad \text{(B34)}
$$

$$
\hat{C}_t^R = \frac{1}{c_y}\frac{2\nu-1}{D}\hat{Y}_t^R - \frac{1-c_y}{c_y}\frac{2\nu-1}{D}\hat{G}_t^R + \frac{2\theta\nu(1-\nu)}{D}\hat{m}_t + \frac{D-(2\nu-1)^2}{\rho D}\hat{Z}_t^R, \qquad \text{(B35)}
$$

$$
\hat{T}_t = \frac{1}{c_y}\frac{2\rho}{D}\hat{Y}_t^R - \frac{1-c_y}{c_y}\frac{2\rho}{D}\hat{G}_t^R - \frac{2\nu-1}{D}\hat{m}_t - \frac{2(2\nu-1)}{D}\hat{Z}_t^R. \qquad \text{(B36)}
$$

Adding (B31) to (B33) yields (B34), subtracting (B33) from (B32) yields (B36), and substituting (B36) into the risk-sharing condition yields (B35).

One can express these equations as the efficient gap terms by subtracting the corresponding

[5] We normalize the function by c_y.

sticky-price equilibrium and the efficient equilibrium equations:

$$\tilde{C}_t^W = \frac{1}{c_y}\tilde{Y}_t^W - \frac{1-c_y}{c_y}\tilde{G}_t^W, \tag{B37}$$

$$\tilde{C}_t^R = \frac{1}{c_y}\frac{2\nu-1}{D}\tilde{Y}_t^R - \frac{1-c_y}{c_y}\frac{2\nu-1}{D}\tilde{G}_t^R + \frac{2\theta\nu(1-\nu)}{D}\hat{m}_t, \tag{B38}$$

$$\tilde{T}_t = \frac{1}{c_y}\frac{2\rho}{D}\tilde{Y}_t^R - \frac{1-c_y}{c_y}\frac{2\rho}{D}\tilde{G}_t^R - \frac{2\nu-1}{D}\hat{m}_t. \tag{B39}$$

The resource constraint for each country can written as follows:

$$Y_{H,t} = \nu\Delta_{H,t}\left(\frac{P_{H,t}}{P_t}\right)^{-\theta}C_t + (1-\nu)\Delta_{H,t}^*\left(\frac{P_{H,t}^*}{P_t^*}\right)^{-\theta}C_t^* + \Delta_{H,t}G_t,$$

$$Y_{F,t} = (1-\nu)\Delta_{F,t}\left(\frac{P_{F,t}}{P_t}\right)^{-\theta}C_t + \nu\Delta_{F,t}^*\left(\frac{P_{F,t}^*}{P_t^*}\right)^{-\theta}C_t^* + \Delta_{F,t}^*G_t^*,$$

where $\Delta_{H,t} \equiv \int_0^1 \left(\frac{P_{H,t}(i)}{P_{H,t}}\right)^{-\sigma} di$, $\Delta_{H,t}^* \equiv \int_0^1 \left(\frac{P_{H,t}^*(i)}{P_{H,t}^*}\right)^{-\sigma} di$, $\Delta_{F,t} \equiv \int_0^1 \left(\frac{P_{F,t}(i)}{P_{F,t}}\right)^{-\sigma} di$ and $\Delta_{F,t}^* \equiv \int_0^1 \left(\frac{P_{F,t}^*(i)}{P_{F,t}^*}\right)^{-\sigma} di$ are the price dispersions for the Home and the Foreign goods, respectively. By approximating the resource constraints in a second order and taking a summation of them, we derive the following:

$$\begin{aligned}
2\hat{C}_t^W - \frac{2}{c_y}\hat{Y}_t^W + \frac{2(1-c_y)}{c_y}\hat{G}_t^W = &-\left[\left(\hat{C}_t^W\right)^2 + \left(\hat{C}_t^R\right)^2\right] + \frac{1}{c_y}\left[\left(\hat{Y}_t^W\right)^2 + \left(\hat{Y}_t^R\right)^2\right] \\
&- \frac{1-c_y}{c_y}\left[\left(\hat{G}_t^W\right)^2 + \left(\hat{G}_t^R\right)^2\right] - \theta\nu(1-\nu)\hat{T}_t^2 \\
&- \frac{1}{c_y}\frac{\sigma}{2\kappa}\left\{[1-(1-\nu)c_y]\pi_{H,t}^2 + c_y(1-\nu)(\pi_{H,t}^*)^2\right. \\
&\left.+[1-(1-\nu)c_y](\pi_{F,t}^*)^2 + c_y(1-\nu)\pi_{F,t}^2\right\}.
\end{aligned} \tag{B40}$$

Then, by subtracting the corresponding efficient equilibrium equation from the equation (B40), we get:

$$\begin{aligned}
2\tilde{C}_t^W - \frac{2}{c_y}\tilde{Y}_t^W + \frac{2(1-c_y)}{c_y}\tilde{G}_t^W = &-\left[\left(\tilde{C}_t^W\right)^2 + \left(\tilde{C}_t^R\right)^2\right] + \frac{1}{c_y}\left[\left(\tilde{Y}_t^W\right)^2 + \left(\tilde{Y}_t^R\right)^2\right] \\
&- \frac{1-c_y}{c_y}\left[\left(\tilde{G}_t^W\right)^2 + \left(\tilde{G}_t^R\right)^2\right] - \theta\nu(1-\nu)\tilde{T}_t^2 \\
&- \left(\hat{C}_t^{fb,W}\tilde{C}_t^W + \hat{C}_t^{fb,R}\tilde{C}_t^R\right) + \frac{2}{c_y}\left(\hat{Y}_t^{fb,W}\tilde{Y}_t^W + \hat{Y}_t^{fb,R}\tilde{Y}_t^R\right) \\
&- \frac{2(1-c_y)}{c_y}\left(\hat{G}_t^{fb,W}\tilde{G}_t^W + \hat{G}_t^{fb,R}\tilde{G}_t^R\right) - 2\theta\nu(1-\nu)\hat{T}^{fb}\tilde{T}_t \\
&- \frac{1}{c_y}\frac{\sigma}{2\kappa}\left\{[1-(1-\nu)c_y]\pi_{H,t}^2 + c_y(1-\nu)(\pi_{H,t}^*)^2\right. \\
&\left.+[1-(1-\nu)c_y](\pi_{F,t}^*)^2 + c_y(1-\nu)\pi_{F,t}^2\right\}.
\end{aligned} \tag{B41}$$

Equations (B21), (B22), (B23), (B37), (B38), (B39), and (B41) will be used in the next subsection.

B.5.4 Further Manipulation of the Welfare Loss Function (LCP)

We use the second-order approximations of the resource constraints to fully approximate the welfare loss function into the second-order terms. By substituting the Equation (B41) into (B14), we obtain:

$$
\begin{aligned}
2\left(\mathcal{L}_t-\mathcal{L}_t^{fb}\right)=&-2\rho\hat{C}_t^{R,fb}\tilde{C}_t^R+2\hat{Z}_t^R\tilde{C}_t^R-2\theta\nu(1-\nu)\hat{T}^{fb}\tilde{T}_t\\
&-\frac{2}{c_y}\eta\hat{Y}_t^{R,fb}\tilde{Y}_t^R-2\frac{1-c_y}{c_y}\rho\hat{G}_t^{R,fb}\tilde{G}_t^R\\
&-2\rho\hat{C}_t^{W,fb}\tilde{C}_t^W+2\hat{Z}_t^W\tilde{C}_t^W\\
&-\frac{2}{c_y}\eta\hat{Y}_t^{W,fb}\tilde{Y}_t^W-2\frac{1-c_y}{c_y}\rho\hat{G}_t^{W,fb}\tilde{G}_t^W\\
&-\rho\left[\left(\tilde{C}_t^W\right)^2+\left(\tilde{C}_t^R\right)^2\right]-\frac{1}{c_y}\eta\left[\left(\tilde{Y}_t^W\right)^2+\left(\tilde{Y}_t^R\right)^2\right]\\
&-\frac{1-c_y}{c_y}\rho\left[\left(\tilde{G}_t^W\right)^2+\left(\tilde{G}_t^R\right)^2\right]-\theta\nu(1-\nu)\tilde{T}_t^2\\
&-\frac{1}{c_y}\frac{\sigma}{2\kappa}\left\{[1-(1-\nu)c_y]\pi_{H,t}^2+c_y(1-\nu)(\pi_{H,t}^*)^2\right.\\
&\qquad\left.+[1-(1-\nu)c_y](\pi_{F,t}^*)^2+c_y(1-\nu)\pi_{F,t}^2\right\}+t.i.p.
\end{aligned}
\tag{B42}
$$

To recast the welfare loss in terms of output gaps, fiscal gaps, relative price gaps, currency misalignment, and inflation, we eliminate the interaction terms between the relative flexible variables ($\hat{X}_t^{R,fb}$) and the relative gap variables ($\tilde{X}_t^R$). By using the relationship $\tilde{C}_t^R=\frac{2\nu-1}{2\rho}\tilde{T}_t+\frac{1}{2\rho}\hat{m}_t$[6] and the equation (B10), we can rearrange the first two rows of the equation (B42) as follows:

$$
\begin{aligned}
2\left(\mathcal{L}_t-\mathcal{L}_t^{fb}\right)=&-\left\{(2\nu-1)\hat{C}_t^{R,fb}+2\theta\nu(1-\nu)\hat{T}_t^{fb}-\frac{2\nu-1}{\rho}\hat{Z}_t^R\right\}\tilde{T}_t\\
&-\left\{\hat{C}_t^{R,fb}-\frac{1}{\rho}\hat{Z}_t^R\right\}\hat{m}_t\\
&-2\eta\hat{Y}_t^{R,fb}\left(\frac{1}{c_y}\tilde{Y}_t^R-\frac{1-c_y}{c_y}\tilde{G}_t^R\right)\\
&-2\rho\hat{C}_t^{W,fb}\tilde{C}_t^W+2\hat{Z}_t^W\tilde{C}_t^W\\
&-\frac{2}{c_y}\eta\hat{Y}_t^{W,fb}\tilde{Y}_t^W-2\frac{1-c_y}{c_y}\rho\hat{G}_t^{W,fb}\tilde{G}_t^W\\
&-\rho\left[\left(\tilde{C}_t^W\right)^2+\left(\tilde{C}_t^R\right)^2\right]-\frac{1}{c_y}\eta\left[\left(\tilde{Y}_t^W\right)^2+\left(\tilde{Y}_t^R\right)^2\right]\\
&-\frac{1-c_y}{c_y}\rho\left[\left(\tilde{G}_t^W\right)^2+\left(\tilde{G}_t^R\right)^2\right]-\theta\nu(1-\nu)\tilde{T}_t^2\\
&-\frac{1}{c_y}\frac{\sigma}{2\kappa}\left\{[1-(1-\nu)c_y]\pi_{H,t}^2+c_y(1-\nu)(\pi_{H,t}^*)^2\right.\\
&\qquad\left.+[1-(1-\nu)c_y](\pi_{F,t}^*)^2+c_y(1-\nu)\pi_{F,t}^2\right\}+t.i.p.
\end{aligned}
$$

[6] This relationship can be easily derived from the equations (B38) and (B39).

Then, substituting the equations (B22) and (B23) into the first row, and the equation (B39) into the third row yields:

$$
\begin{aligned}
2\left(\mathcal{L}_t - \mathcal{L}_t^{fb}\right) = & -\left\{\frac{1}{c_y}\hat{Y}_t^{R,fb} - \frac{1-c_y}{c_y}\hat{G}_t^{R,fb} - \frac{2\nu-1}{\rho}\hat{Z}_t^R\right\}\tilde{T}_t \\
& -\left\{\hat{C}_t^{R,fb} - \frac{1}{\rho}\hat{Z}_t^R\right\}\hat{m}_t \\
& -\frac{\eta}{\rho}D\hat{Y}_t^{R,fb}\left\{\tilde{T}_t + \frac{2\nu-1}{D}\hat{m}_t\right\} \\
& -2\rho\hat{C}_t^{W,fb}\tilde{C}_t^W + 2\hat{Z}_t^W\tilde{C}_t^W \\
& -\frac{2}{c_y}\eta\hat{Y}_t^{W,fb}\tilde{Y}_t^W - 2\frac{1-c_y}{c_y}\rho\hat{G}_t^{W,fb}\tilde{G}_t^W \\
& -\rho\left[\left(\tilde{C}_t^W\right)^2 + \left(\tilde{C}_t^R\right)^2\right] - \frac{1}{c_y}\eta\left[\left(\tilde{Y}_t^W\right)^2 + \left(\tilde{Y}_t^R\right)^2\right] \\
& -\frac{1-c_y}{c_y}\rho\left[\left(\tilde{G}_t^W\right)^2 + \left(\tilde{G}_t^R\right)^2\right] - \theta\nu(1-\nu)\tilde{T}_t^2 \\
& -\frac{1}{c_y}\frac{\sigma}{2\kappa}\left\{[1-(1-\nu)c_y]\pi_{H,t}^2 + c_y(1-\nu)(\pi_{H,t}^*)^2\right. \\
& \qquad \left. + [1-(1-\nu)c_y](\pi_{F,t}^*)^2 + c_y(1-\nu)\pi_{F,t}^2\right\} + t.i.p.
\end{aligned}
$$

Using the equation (B1), (B2), (B23) and the relation between the flexible relative price and the flexible relative output gap (B11) rids out the first three rows.

$$
\begin{aligned}
2\left(\mathcal{L}_t - \mathcal{L}_t^{fb}\right) = & -2\rho\hat{C}_t^{W,fb}\tilde{C}_t^W + 2\hat{Z}_t^W\tilde{C}_t^W \\
& -\frac{2}{c_y}\eta\hat{Y}_t^{W,fb}\tilde{Y}_t^W - 2\frac{1-c_y}{c_y}\rho\hat{G}_t^{W,fb}\tilde{G}_t^W \\
& -\rho\left[\left(\tilde{C}_t^W\right)^2 + \left(\tilde{C}_t^R\right)^2\right] - \frac{1}{c_y}\eta\left[\left(\tilde{Y}_t^W\right)^2 + \left(\tilde{Y}_t^R\right)^2\right] \\
& -\frac{1-c_y}{c_y}\rho\left[\left(\tilde{G}_t^W\right)^2 + \left(\tilde{G}_t^R\right)^2\right] - \theta\nu(1-\nu)\tilde{T}_t^2 \\
& -\frac{1}{c_y}\frac{\sigma}{2\kappa}\left\{[1-(1-\nu)c_y]\pi_{H,t}^2 + c_y(1-\nu)(\pi_{H,t}^*)^2\right. \\
& \qquad \left. + [1-(1-\nu)c_y](\pi_{F,t}^*)^2 + c_y(1-\nu)\pi_{F,t}^2\right\} + t.i.p.
\end{aligned}
$$

To substitute out the interaction terms between the world first-best variables ($\hat{X}_t^{W,fb}$) and the world gap variables ($\tilde{X}_t^W$), we first use the equation (B7), (B21) and (B37) to yield:

$$
\begin{aligned}
2\left(\mathcal{L}_t - \mathcal{L}_t^{fb}\right) =& 2\eta \hat{Y}_t^{W,fb}\left(\frac{1}{c_y}\tilde{Y}_t^W - \frac{1-c_y}{c_y}\tilde{G}_t^W\right) \\
& - \frac{2}{c_y}\eta \hat{Y}_t^{W,fb}\tilde{Y}_t^W - 2\frac{1-c_y}{c_y}\rho \hat{G}_t^{W,fb}\tilde{G}_t^W \\
& - \rho\left[\left(\tilde{C}_t^W\right)^2 + \left(\tilde{C}_t^R\right)^2\right] - \frac{1}{c_y}\eta\left[\left(\tilde{Y}_t^W\right)^2 + \left(\tilde{Y}_t^R\right)^2\right] \\
& - \frac{1-c_y}{c_y}\rho\left[\left(\tilde{G}_t^W\right)^2 + \left(\tilde{G}_t^R\right)^2\right] - \theta\nu(1-\nu)\tilde{T}_t^2 \\
& - \frac{1}{c_y}\frac{\sigma}{2\kappa}\left\{[1-(1-\nu)c_y]\pi_{H,t}^2 + c_y(1-\nu)(\pi_{H,t}^*)^2\right. \\
& \qquad \left. + [1-(1-\nu)c_y](\pi_{F,t}^*)^2 + c_y(1-\nu)\pi_{F,t}^2\right\} + t.i.p.
\end{aligned}
$$

Then, using the equation (B10) will substitute out the interaction terms:

$$
\begin{aligned}
2\left(\mathcal{L}_t - \mathcal{L}_t^{fb}\right) =& -\rho\left[\left(\tilde{C}_t^W\right)^2 + \left(\tilde{C}_t^R\right)^2\right] - \frac{1}{c_y}\eta\left[\left(\tilde{Y}_t^W\right)^2 + \left(\tilde{Y}_t^R\right)^2\right] \\
& - \frac{1-c_y}{c_y}\rho\left[\left(\tilde{G}_t^W\right)^2 + \left(\tilde{G}_t^R\right)^2\right] - \theta\nu(1-\nu)\tilde{T}_t^2 \\
& - \frac{1}{c_y}\frac{\sigma}{2\kappa}\left\{[1-(1-\nu)c_y]\pi_{H,t}^2 + c_y(1-\nu)(\pi_{H,t}^*)^2\right. \\
& \qquad \left. + [1-(1-\nu)c_y](\pi_{F,t}^*)^2 + c_y(1-\nu)\pi_{F,t}^2\right\} + t.i.p.
\end{aligned}
$$

Lastly, substituting $\tilde{C}_t^W$, $\tilde{C}_t^R$, and $\tilde{T}_t$ using the equations (B37), (B38), and (B39), and some rearrangement, we can derive the welfare loss function as follows[7]:

$$
\begin{aligned}
2\left(\mathcal{L}_t - \mathcal{L}_t^{fb}\right) =& -\left(\frac{\rho}{c_y} + \eta\right)\left(\tilde{Y}_t^W\right)^2 - \left(\frac{\rho}{c_y D} + \eta\right)\left(\tilde{Y}_t^R\right)^2 \\
& - \rho\frac{1-c_y}{c_y}\left(\tilde{G}_t^W\right)^2 - \rho(1-c_y)\left(\frac{1-c_y}{c_y D} + 1\right)\left(\tilde{G}_t^R\right)^2 \\
& + 2\rho\frac{1-c_y}{c_y}\tilde{Y}_t^W\tilde{G}_t^W + 2\rho\frac{1-c_y}{c_y D}\tilde{Y}_t^R\tilde{G}_t^R \\
& - \frac{c_y\theta\nu(1-\nu)}{D}\hat{m}_t^2 \\
& - \frac{\sigma}{\kappa}\left\{\left(\hat{\pi}_t^W\right)^2 + \left(\hat{\pi}_{cpi,t}^R\right)^2 - 2\left(1-c_y\right)(1-v)\hat{\pi}_{cpi,t}^R\Delta\hat{T}_t\right. \\
& \left. + \left[c_y(2v-1) + (1-v)\right](1-v)\left(\Delta\hat{T}_t\right)^2\right\}. \qquad \text{(B43)}
\end{aligned}
$$

[7] We normalize the function by c_y.

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